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# SECOND-ORDER EFFECTS AND SAINT VENANT'S PRINCIPLE IN THE TORSION PROBLEM OF A NONLINEAR ELASTIC ROD

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*The torsion problem of a circular nonlinear elastic rod loaded by end moments is considered. The solution constructed by the method of successive approximations taking into account second-order effects is compared with the solution obtained by a semi-inverse method. It is shown that the dead-loading assumption breaks the symmetry of the Cauchy stress tensor in a certain region. A refined formulation of Saint Venant's principle is proposed for the problem of determining integral strain characteristics.*

**Key words:** *Pointing effect, semi-inverse method, second-order effects, Saint Venant's principle.*

**Introduction.** In designing many current high-precision devices, it is necessary to take into account the effects of physical and geometrical nonlinearities. (For example, in designing and calibrating a rod dynamometer, one should take into account the Pointing effect — the elongation of the rod due to torsion.)

Analysis of this problem reduces to determining second-order effects in the torsion problem of a nonlinear elastic circular rod loaded by end moments. This classical problem has been the subject of many studies, among which we mention the work of Lur'e. In [1], he proposed a method of successive approximations for determining second-order effects in the deformation of bodies of various shapes. In [2], it is shown that the axial elongation of a cylinder obtained by the method of successive approximations [1] differs from that obtained by the semi-inverse method. In the present paper, it is shown that the reason for this difference is that although the integral characteristics of the external load (axial force and torque) are equal in these problems, the solutions constructed correspond to different force distributions over the end surfaces of the cylinder. Moreover, the effect of this difference on the integral strain characteristic — cylinder elongation — is discussed.

**Method of Successive Approximations.** The essence of the method described in [1] is that the problem of the equilibrium of a nonlinear elastic body of the form

$$\overset{0}{\nabla} \cdot D + \rho_0 \boldsymbol{k} = 0; \tag{1}$$

$$\boldsymbol{n} \cdot D \, do = \boldsymbol{f} \, dO, \tag{2}$$

where $\overset{0}{\nabla}$ is the gradient operator for the reference configuration, $\rho_0$ is the density of the body in the reference configuration, $\boldsymbol{k}$ is the mass-force vector, $\boldsymbol{n}$ is the outward normal vector to the surface of the body, $do$ and $dO$ are elementary areas of the surface in the reference and current configurations, respectively, $D$ is the Piola stress tensor, and $\boldsymbol{f}$ is the external dead load normalized by the deformed surface,

$$\boldsymbol{f} \, dO = \boldsymbol{f}^0 \, do, \tag{3}$$

is replaced by the succession of two problems:

— the linear problem

$$\overset{0}{\nabla} \cdot \sigma(\boldsymbol{v}) + \rho_0 \boldsymbol{k} = 0, \qquad \boldsymbol{n} \cdot \sigma(\boldsymbol{v}) = \boldsymbol{f}^0; \tag{4}$$

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— the problem of second-order effects

$$\overset{0}{\nabla}\cdot\sigma(\boldsymbol{w})+\rho_0\boldsymbol{k}_*=0,\qquad \boldsymbol{n}\cdot\sigma(\boldsymbol{w})=\boldsymbol{f}_* \tag{5}$$

based on the previous problem. Here $\boldsymbol{v}$ is the linear displacement vector in the solution of the linear problem, $\boldsymbol{w}$ is the quadratic displacement vector in the solution of the problem of second-order effects, $\boldsymbol{k}_*$ and $\boldsymbol{f}_*$ are the mass- and surface-force vectors, respectively, in the problem of second-order effects, and $\sigma$ is the Cauchy stress tensor.

In (4) and (5), dependence of the tensor $\sigma$ on the displacement vector corresponds to the classical law of the linear theory of elasticity:

$$\sigma(\boldsymbol{u})=\lambda\overset{0}{\nabla}\cdot\boldsymbol{u}E+\mu(\overset{0}{\nabla}\boldsymbol{u}+\overset{0}{\nabla}\boldsymbol{u}^{\mathrm{t}}),$$

and the mass and surface forces are represented by the vectors

$$\rho_0\boldsymbol{k}_*=\overset{0}{\nabla}\cdot\Big(\overset{0}{\nabla}\boldsymbol{v}\cdot\sigma(\boldsymbol{v})+\sigma'(\boldsymbol{v})\Big),\qquad \boldsymbol{f}_*=-\boldsymbol{n}\cdot\Big(\overset{0}{\nabla}\boldsymbol{v}\cdot\sigma(\boldsymbol{v})+\sigma'(\boldsymbol{v})\Big).$$

Problems (4) and (5) are obtained from (1) and (2), respectively, by decomposition of the displacement vector $\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{w}$ ($\boldsymbol{v}$ is the presumably known solution of the linear problem and $\boldsymbol{w}$ compensates for second-order terms) and corresponding decomposition of the Piola stress tensor:

$$D(\boldsymbol{u})=\sigma(\boldsymbol{v})+\sigma(\boldsymbol{w})+\overset{0}{\nabla}\boldsymbol{v}\cdot\sigma(\boldsymbol{v})+\sigma'(\boldsymbol{v})$$

($\boldsymbol{u}$ is the displacement vector in the nonlinear problem and $\sigma'$ is the quadratic component in the decomposition of the Piola stress tensor).

The expression for the tensor $\sigma'$ depends on the form of the nonlinear elastic potential $W$:

$$\sigma'=E\Big[\lambda\Big(\boldsymbol{\omega}\cdot\boldsymbol{\omega}+(1/2)\mathrm{tr}\,(\varepsilon^2)\Big)+\Big((\overset{0}{\nabla}\cdot\boldsymbol{u})^2-\mathrm{tr}\,(\varepsilon^2)\Big)a+b(\overset{0}{\nabla}\cdot\boldsymbol{u})^2\Big]$$

$$+c\varepsilon^2+d\varepsilon^2\,\overset{0}{\nabla}\cdot\boldsymbol{u}+\mu\,\overset{0}{\nabla}\boldsymbol{u}^{\mathrm{t}}\cdot\overset{0}{\nabla}\boldsymbol{u}.$$

Here

$$a=4\Big[\Big(\frac{\partial}{\partial I_2}+\frac{\partial}{\partial I_3}\Big)F\Big]^0;\qquad b=4\Big[\Big(\frac{\partial}{\partial I_1}+2\,\frac{\partial}{\partial I_2}+\frac{\partial}{\partial I_3}\Big)^2F\Big]^0,$$

$$F=\frac{\partial W}{\partial I_1}+(I_1-1)\,\frac{\partial W}{\partial I_2}+I_3\,\frac{\partial W}{\partial I_3},\qquad c=8\,\frac{\partial W^0}{\partial I_3},$$

$$d=-8\Big(\frac{\partial^2 W}{\partial I_1\,\partial I_2}+\frac{\partial W}{\partial I_3}+\frac{\partial^2 W}{\partial I_1\,\partial I_3}+3\frac{\partial^2 W}{\partial I_2\,\partial I_3}+\frac{\partial^2 W}{\partial I_3^2}+2\,\frac{\partial^2 W}{\partial I_2^2}\Big)^0,$$

$\boldsymbol{\omega}=(1/2)\overset{0}{\nabla}\times\boldsymbol{u}$ is the rotation vector, $\varepsilon=(1/2)(\overset{0}{\nabla}\boldsymbol{u}+\overset{0}{\nabla}\boldsymbol{u}^{\mathrm{t}})$ is the linear strain tensor, $I_k(G)$ are the principal invariants of the Cauchy strain measure, and $\lambda$ and $\mu$ are the Lamé constants; the superscript 0 refers to the reference configuration, i.e., after the differentiation, one should set $I_1=I_2=3$ and $I_3=1$.

In the case of Murnaghan materials, whose strain energy is given by the function

$$W=(1/2)(\lambda+2\mu)j_1^2-2\mu j_2+(1/3)(l+2m)j_1^3-2mj_1j_2+nj_3$$

[$j_1=\mathrm{tr}\,K$, $j_2=(1/2)(\mathrm{tr}^2K-\mathrm{tr}\,K^2)$, $j_3=\det K$, and $K=(1/2)(G-E)$ is the Cauchy strain tensor], the constants appearing in the tensor $\sigma'$ become

$$a=-m+n/2,\qquad b=l,\qquad c=n,\qquad d=-n+2m.$$

In a number of cases, the formulation (5) allows one to calculate some strain characteristics without determining the vector $\boldsymbol{w}$, i.e., without solving the boundary-value problem. For the torsion problem, this characteristic is the axial elongation of the cylinder. (The axial elongation of the cylinder in the absence of axial traction is known as the Pointing effect.) In [1], the elongation was found using the following approach. In the general case of an elastic isotropic body, the force tensor is calculated for the problem (5):

$$B = \iiint\limits_V \rho_0 \boldsymbol{k}_* \boldsymbol{R}\, dV + \iint\limits_S \boldsymbol{f}_* \boldsymbol{R}\, dS$$

$$= \iiint\limits_V \rho_0 \boldsymbol{k}_* \boldsymbol{R}\, dV - \iint\limits_S \boldsymbol{n} \cdot \Big( \overset{0}{\nabla} \boldsymbol{v} \cdot \sigma(\boldsymbol{v}) + \sigma'(\boldsymbol{v}) \Big) \boldsymbol{R}\, dS$$

$$= \iiint\limits_V \overset{0}{\nabla} \cdot \Big( \overset{0}{\nabla} \boldsymbol{v} \cdot \sigma(\boldsymbol{v}) + \sigma'(\boldsymbol{v}) \Big) \boldsymbol{R}\, dV - \iiint\limits_V \overset{0}{\nabla} \cdot \Big( \overset{0}{\nabla} \boldsymbol{v} \cdot \sigma(\boldsymbol{v}) + \sigma'(\boldsymbol{v}) \Big) \boldsymbol{R}\, dV$$

$$- \iiint\limits_V \Big( \overset{0}{\nabla} \boldsymbol{v} \cdot \sigma(\boldsymbol{v}) + \sigma'(\boldsymbol{v}) \Big)^{\mathrm{t}} \cdot \overset{0}{\nabla} \boldsymbol{R}\, dV$$

($S$ and $V$ are the area and volume of the body in the current configuration and $\boldsymbol{R}$ is the radius vector of the point in the current configuration). After some manipulations, the relation for the force tensor becomes

$$B = - \iiint\limits_V \Big( \sigma(\boldsymbol{v}) \cdot \overset{0}{\nabla} \boldsymbol{v}^{\mathrm{t}} + \sigma'(\boldsymbol{v}) \Big)\, dV.$$

The mean value of the stress tensor $\sigma(\boldsymbol{w})$ is given by

$$\sigma_m(\boldsymbol{w}) = \frac{1}{V} \iiint\limits_V \sigma(\boldsymbol{w})\, dV = \frac{1}{V}\, B = -\frac{1}{V} \iiint\limits_V \Big( \sigma(\boldsymbol{v}) \cdot \overset{0}{\nabla} \boldsymbol{v}^{\mathrm{t}} + \sigma'(\boldsymbol{v}) \Big)\, dV,$$

and the mean value of the linear strain tensor is given by

$$2\mu \varepsilon_m(\boldsymbol{w}) = \sigma_m(\boldsymbol{w}) - E\, \frac{\lambda}{3\lambda + 2\mu}\, I_1(\sigma_m(\boldsymbol{w})).$$

The elongation per unit length of an elastic circular rod is determined from the expression

$$\Delta L / L = \boldsymbol{e}_z \cdot \varepsilon_m(\boldsymbol{w}) \cdot \boldsymbol{e}_z$$

($\boldsymbol{e}_z$ is the unit coordinate vector in the reference configuration) and for Murnaghan materials, it is calculated by the formula

$$\frac{\Delta L}{L} = \frac{\psi^2 r_1^2}{4} \Big( \frac{1}{3\lambda + 2\mu} \Big( \lambda - \frac{n\lambda}{4\mu} - m \Big) - 1 \Big), \tag{6}$$

where $\psi$ is the rotation angle per unit length, $r_1$ is the radius of the cylinder, and $n$ and $m$ are the Murnaghan material constants.

**Solution of the Problem by the Semi-Inverse Method.** An analysis of the torsion problem (1), (2) for a circular rod using the semi-inverse method is based on transforming the reference (undeformed) configuration to a current (deformed) configuration of the form

$$R = R(r), \qquad \Phi = \varphi + \psi z, \qquad Z = \alpha z,$$

where $\alpha$ is a parameter that expresses the rod elongation and $r$, $\varphi$, and $z$ are cylindrical coordinates in the reference configuration. In this case, the equilibrium equations and the boundary conditions of the stress-free lateral surface are satisfied exactly, whereas the boundary conditions on the end surfaces are satisfied in an integral sense, which implies that the axial tensile force vanishes and the resultant moment of the end stresses is equal to the specified torque. As a result, in particular, one obtains the parameters $\psi$ and $\alpha$ and, hence, the axial elongation $\alpha - 1$.

For Murnaghan materials, the equation for $R(r)$ is rather cumbersome and is not given here. However, if we retain terms not higher than the second order in this equation, i.e., if we write the unknown function $R(r)$ as

$$R(r) = r + \psi^2 f(r) + \dots\ ,$$

the linear equation for $f(r)$ can be solved explicitly. The elongation obtained from this equation with accuracy up to second-order terms is given by

$$\frac{\Delta L}{L} = \frac{\psi^2 r_1^2}{4} \Big( \frac{1}{3\lambda + 2\mu} \Big( 2\lambda - \frac{n\lambda}{4\mu} - m \Big) - 1 \Big)$$

and differs from (6) in the factor at the parameter $\lambda$ in parentheses. Since the elongation obtained above differs greatly from that given by (6), it is important to find the reason for this difference.

The further analysis is based on the exact solution of the nonlinear rod torsion problem, which can be obtained for the simplified Blatz–Ko material model, one of the models of compressible nonlinear elastic media. In this case, the elastic potential $W$ becomes

$$W = (1/2)\mu(I_2/I_3 + 2\sqrt{I_3} - 5).$$

For Blatz–Ko materials, the Piola stress tensor is written as

$$D = (\mu/I_3)(I_1 E - G + (I_3^{3/2} - I_2)G^{-1}) \cdot C,$$

and the boundary-value problem for determining the unknown function of the radius $R(r)$ is given by

$$\begin{gathered} R''(r) = \frac{R'(r)R(r)^3 - r^3(R'(r))^4}{3R(r)^3 r}, \\ R(0) = 0, \qquad -(R'(r_1))^3 R(r_1)\alpha + r_1 = 0, \end{gathered} \tag{7}$$

where $r_1$ is the outer radius of the rod.

The boundary-value problem (7) has the analytical solution

$$R(r) = \alpha^{-1/4} r.$$

The dependence $\alpha(\psi)$ is found using the condition of zero axial force

$$Q = \iint\limits_S \sigma_{zz}\,ds = 2\pi \int\limits_0^{r_1} \sigma_z R R'(r)\,dr = 2\pi\Big(-\frac{1}{4}\,\frac{\mu\psi^2 r_1^4}{\alpha^3} + \frac{1}{2}\,r_1^2\mu\,\frac{1-\alpha^{5/2}}{\alpha^3}\Big)$$

($\sigma_{zz}$ are the normal stresses at the end of the rod). Thus, we obtain

$$\alpha(\psi) = (1/4)(16\psi^2 r_1^2 + 32)^{2/5} \approx 1 + (1/5)\psi^2 r_1^2.$$

Consequently, the elongation per unit length is calculated by the formula

$$(\alpha L - L)/L = \alpha - 1 = (1/5)\psi^2 r_1^2.$$

Establishing the relation between the Murnaghan and Blatz–Ko material constants up to second-order terms in the form

$$n = -8\mu, \qquad m = -5\mu, \qquad l = -(1/2)\mu,$$

for Blatz–Ko materials, formula (6) can be written as $\Delta L/L = (3/20)\psi^2 r_1^2$, whereas the elongation obtained by the semi-inverse method is given by $\Delta L/L = (1/5)\psi^2 r_1^2$. Thus, the values of the elongation per unit length obtained using the approaches described above differ by 25%.

The exact solution of the nonlinear problem given above allows one to clarify the reason why the two approaches give different results.

For the torsion problem, the equations and boundary conditions (5) become

$$\begin{gathered} \overset{0}{\nabla} \cdot \sigma(\boldsymbol{w}_1) + \rho_0 \boldsymbol{k}_* = 0, \\ \boldsymbol{e}_r \cdot \sigma(\boldsymbol{w}_1) = -2z^2\mu\psi^2 \boldsymbol{e}_r - zr\mu\psi^2 \boldsymbol{e}_z \quad \text{for} \quad r = r_1, \\ \boldsymbol{e}_z \cdot \sigma(\boldsymbol{w}_1) = \mu\psi^2(r^2 - z^2)\boldsymbol{e}_z \quad \text{for} \quad z = \pm L/2 \end{gathered} \tag{8}$$

($L$ is the cylinder length). Using the expression for the additional vector $\boldsymbol{w}_2$ obtained by the semi-inverse method, we write this problem as

$$\begin{gathered} \overset{0}{\nabla} \cdot \sigma(\boldsymbol{w}_2) + \rho_0 \boldsymbol{k}_* = 0, \\ \boldsymbol{e}_r \cdot \sigma(\boldsymbol{w}_2) = -2z^2\mu\psi^2 \boldsymbol{e}_r - zr\mu\psi^2 \boldsymbol{e}_z \quad \text{for} \quad r = r_1, \\ \boldsymbol{e}_z \cdot \sigma(\boldsymbol{w}_2) = -zr\mu\psi^2 \boldsymbol{e}_r - \mu\psi^2(z^2 - 1/2r_1^2)\boldsymbol{e}_z \quad \text{for} \quad z = \pm L/2. \end{gathered} \tag{9}$$

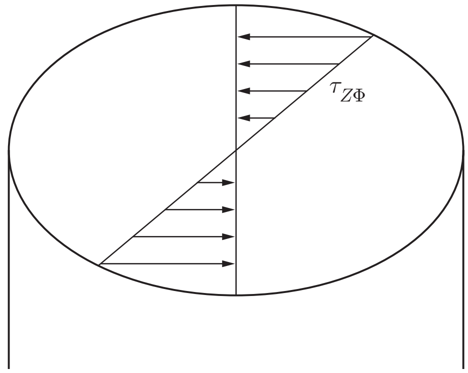


Fig. 1. Shear-stress distribution for the natural boundary condition at the end of the cylinder.

From (8) and (9) it follows that the stress fields on the end surfaces differ, and this causes the discrepancy between the results.

It is worth noting that the problems (8) and (9) are formulated in the coordinates of the reference configuration used in [1].

**Influence of the Discrepancy in the End Stresses on the Solution.** To estimate the effect of the boundary conditions on the specific elongation, we consider the following problem formulated as the difference between the linear problems of second-order effects (8) and (9) obtained by the approaches considered above:

$$\begin{gathered} \overset{0}{\nabla} \cdot \sigma(\boldsymbol{\varsigma}) = 0, \\ \boldsymbol{e}_r \cdot \sigma(\boldsymbol{\varsigma}) = 0, \qquad \boldsymbol{e}_z \cdot \sigma(\boldsymbol{\varsigma}) = -zr\mu\psi^2 \boldsymbol{e}_r - \mu\psi^2(r^2 - (1/2)r_1^2)\boldsymbol{e}_z \quad \text{for} \quad z = \pm L/2. \end{gathered} \tag{10}$$

Here $\boldsymbol{\varsigma} = \boldsymbol{w}_2 - \boldsymbol{w}_1$.

Problem (10) has a serious drawback: the boundary conditions are in contradiction to the symmetry condition for the stress tensor on the circles bounding the ends of the cylinder and, hence, they make the tensor asymmetric in a certain region enclosing these circles.

Indeed, from (8) it follows that by virtue of the end boundary conditions, the shear stresses along the boundary circle are given by

$$\tau_{zr} = -\mu\, r_1 \psi^2 L,$$

and the boundary conditions on the lateral surface imply that

$$\tau_{rz} \equiv 0.$$

Thus, the stress tensor in the problem (8) becomes asymmetric due to the dead-loading assumption (3).

Indeed, in the derivation of the boundary-value problem of second-order effects in the starting nonlinear formulation, if the boundary condition of the type of (2), (3), which refers to dead loading, is replaced by the reasonably natural boundary condition

$$\boldsymbol{f} = \mu r \psi \boldsymbol{e}_\Phi \tag{11}$$

(Fig. 1), then after converting from the coordinates of the current configuration to the coordinates of the reference configuration used in the problem, we find that the problem becomes symmetric and its solution is identical to that obtained by the semi-inverse method.

It should be noted that in the problem of a rod under uniform tensile load, the method proposed by Lur'e and the semi-inverse method give identical results. The dead-loading assumption is natural for the tension problem, but it is not physically justified for the torsion problem. Moreover, the example of the load (11) shows the mathematical incorrectness of this assumption, which breaks the symmetry of the stress tensor. Indeed, in this case, relation (3) leads to the boundary condition

$$\boldsymbol{n} \cdot \sigma(\boldsymbol{v}) = \boldsymbol{f}^0$$

or explicitly

$$\boldsymbol{n} \cdot \sigma(\boldsymbol{v}) = \mu r \psi \boldsymbol{e}_\varphi.$$

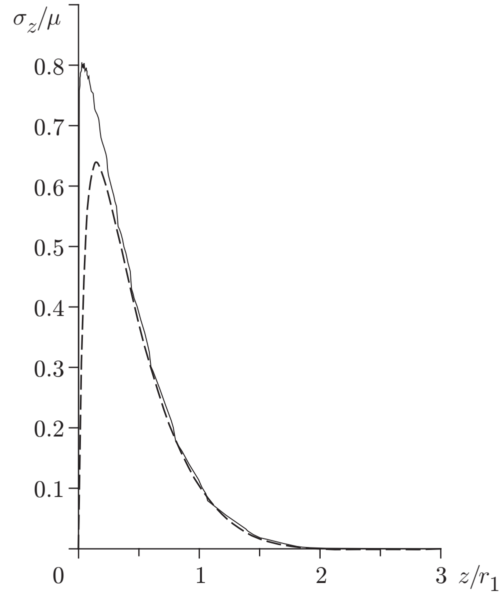


Fig. 2. Distribution of the normal stresses $\sigma_z$ over the lateral surface of the rod, obtained using the finite-element method (solid curve) and the homogeneous-solution method (dashed curve) for $\nu = 0.25$, $L/r_1 = 12$, $\psi r_1 = 0.3$, and $r_1 = 1$.

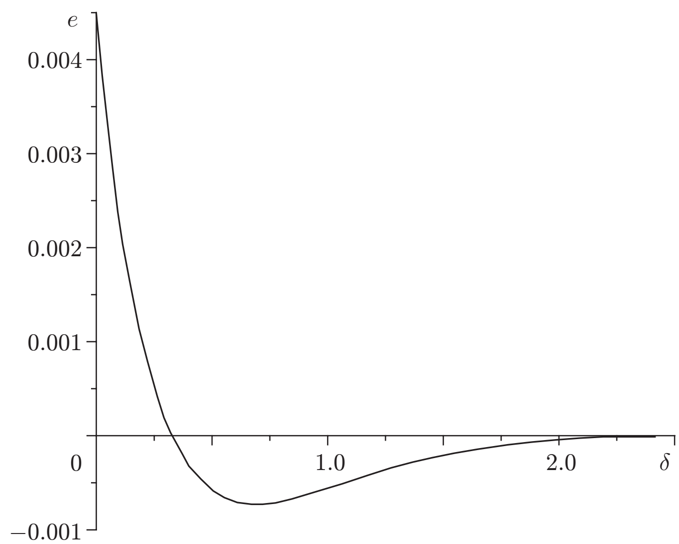


Fig. 3. Elongation per unit length $e$ versus the parameter $\delta$.

However, the quadratic terms in the expression for $\boldsymbol{f}\, dO/do$ have the form $-\mu r\psi^2 z\boldsymbol{e}_r + \boldsymbol{f}^0$ and the boundary conditions at the end are given by

$$\boldsymbol{n} \cdot \sigma(\boldsymbol{w}) = \boldsymbol{f}_* - \mu\, r\psi^2 z\boldsymbol{e}_r.$$

The second term which does not enter the boundary condition is responsible for both the asymmetric tensor and the discrepancy between the values of the axial elongation.

Problem (10) is of interest as a linear elastic problem in which the loads have zero resultants but nonetheless cause axial elongation of the cylinder, which may appear at first sight to be in contradiction with Saint Venant's principle. Below, we show that this is not true.

We modify the problem by replacing the linear representation of the shear stresses $\tau_{zr}$ by a piecewise linear representation that agrees with the symmetry condition of $\sigma$:

$$\tau_{zr} = \begin{cases} \beta r, & r \in [0, r_1 - \varepsilon], \\ (\beta/\varepsilon)(r_1 - r)(r_1 - \varepsilon), & r \in [r_1 - \varepsilon, r_1], \end{cases} \qquad \beta = -(L/2)\psi^2\mu.$$

Figure 2 shows the distribution of the normal stresses $\sigma_z$ over the lateral surface of the rod, obtained using the FlexPDE finite-element package and the homogeneous-solution method [3] with the Maple software.

With distance from the end, the stresses decay rapidly, and at a distance equal to the rod diameter, they almost vanish. This result is supported by calculations performed for various lengths of the cylinder; therefore Saint Venant's principle (stresses vanish far from the region where self-equilibrated loads are applied) holds in this problem.

We consider a cylinder of length $\tilde{L} = L - 2\delta$ located at a distance $\delta$ from the ends of the rod. Figure 3 shows the elongation per unit length of the cylinder $e = \Delta\tilde{L}/\tilde{L}$ as a function of the parameter $\delta$. Using this dependence, we determine a zone whose elongation is negligibly small and, hence, whose elongation per unit length in the starting torsion problem depends only on the integral boundary conditions. Calculations for cylinders of various geometries show that the sought-for zone is the region of the rod for which $\delta/L > 1/6$. The results obtained imply that Saint Venant's principle is applicable to integral strain characteristics for a certain part of the body at a distance from the loaded regions but not for the entire body.

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